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DYNAMICAL ERROR ANALYSIS IN ORBIT DETERMINATION SYSTEMS

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ABSTRACT

In this report, a method is described in which the overall error or uncertainty involved in spacecraft trajectory determinations is used to define a time dependent error bound for the coordinates of the spacecraft. The differences obtained by comparing tracking data with theory reflect errors or uncertainties in the Hamiltonian of the system, and in turn form a type of Canonical ensemble considered in statistical physics. Such considerations then allows one to derive a set of virtual forces that account for the uncertainties in the Hamiltonian which give rise to the calculated differences from the true orbit.

CONTENTS

DYNAMICAL ERROR ANALYSIS IN ORBIT DETERMINATION SYSTEMS

I. INTRODUCTION

There has been much interest recently concerning error analysis of spacecraft trajectory systems. Error analysis can be defined as the ability to describe the effect of inherent uncertainties of an overall trajectory determination system on the computational accuracy of the system. There are several sources responsible for the presence of such uncertainties, the most important being due to the inability to correctly describe the physics of the problem. Hence, the mathematical modelling of the forces involved, and the physical data that is used, provide at best only a good starting point for calculation of an orbit. In addition, the type and quality of the observational data introduce further complications. The usual method of handling such matters is to perform what is called an orbit improvement or differential correction over many revolutions of an orbit, by comparing computed with observed data of the spacecraft and 'correcting' or updating the initial conditions of the differential equations of motion. Here then, the observed data obtained by the tracking systems is considered to be the true or correct orbit data. In reality, the accuracy of the observational data will depend upon the tracking system that is employed. Even after fitting the orbit, the post convergence residuals do not account for uncertainties in coordinates of the tracking sites. Most important, even though the constants of integration may be determined well, the accuracy of the calculated orbit will still depend upon the accuracy of the orbit generator (Reference 1).

In this report, a method is described in which the overall error or uncertainty involved in trajectory calculations is used to define a time dependent error bound for the coordinates of the spacecraft.

II. CONSIDERATIONS OF STATISTICAL PHYSICS

Let a system be characterized by a set of generalized coordinates and their equations of motion. In the canonical form we have,

$$\frac{dP_K}{dt} = -\frac{\partial H}{\partial q_K}, K = 1, \ldots, f \qquad (2.1)$$

$$\frac{dq_K}{dt} = \frac{\partial H}{\partial P_K}$$

where H is the Hamiltonian function. In the theory of gases, statistical thermodynamics introduces the notion of phase space in which the whole gas is represented by a single point with coordinates p and q (Reference 2). Associated with any physical system, is an ensemble of points in phase space, each of which represents a different possible state for the system. This set of points is the Gibbs ensemble and each representative point moves along a curve according to the Hamiltonian equations. It is assumed that such a curve is uniquely determined by a complete set ($q_{01}, \ldots, p_{01}, \ldots$) of initial conditions and therefore these orbits can never intersect. In addition, the points of the Gibbs ensemble contained within a closed hypersurface can never pass through that hypersurface if the surface moves according to Equations (2.1). When special symmetries are present such as when energy is conserved, each representative point is forced to move on a hypersurface or ergodic surface. An important form for the density of representative points corresponding to thermodynamic states is the Canonical ensemble and is given by

$$\rho\,(E) = e^{(\psi - E)/\theta}\,, \tag{2.2}$$

where ψ and θ are two constants characterizing the distribution. This ensemble admits an energy fluxuation in the system (ψ - E), corresponding to a set of non-interacting ergodic surfaces in phase space among which the phase point moves. From this, the expectation value of any observable A is then given by

$$\langle A \rangle = \int A\,\rho\, d\,\tau, \tag{2.3}$$

where $d\tau$ is a volume element in phase space. We shall return to this point in a later section.

In a recent article, (Reference 3) it was shown by similar statistical considerations, that by relating the angular momentum to a quantity defined as the virial parameter through a derived quantity called the Vector Density Function, it is shown that for any conserative system, the periodic variations of an orbit, and consequently the cross track error, will time average to zero.

In this report, we shall attempt to describe the motion of the system point or phase point between ergodic surfaces by letting the energy fluxuation of the Canonical ensemble now describe an 'energy fluxuation' of our orbit system due to the many sources of error inherent in it, that is, since a correct Hamiltonian is not presently possible because of the many difficulties described above, the

system point will be said to occupy any position on any of the 'ergodic surfaces'. These ergodic surfaces then represent energy fluxuations arising from all uncertainties. We only know the position described by (2.3) above given as the most likely by the orbit generator. At this point, we try to make use of observational data which is compared with our corresponding calculated data to define uncertainty on a predicted basis for the orbit system, in terms of our 'Canonical ensemble.'

III. FUNDAMENTAL EQUATIONS

Corresponding to a path described by an artificial earth satellite, the system point will describe such a conic section in phase space governed by equations (2.1). 'Fluxuations' in the motion of the system point and hence the computed orbit can then be described by the following time derivatives of the orbital elements (Reference 4),

$$\frac{da}{dt} = 2\left(\frac{a^3}{\mu}\right)^{1/2} (1 - e^2)^{1/2} \left[R' e \sin v + T' + T' e \cos v\right]$$

$$\frac{de}{dt} = \left(\frac{a}{\mu}\right)^{1/2} (1 - e^2)^{1/2} \left[R' \sin v + T' \cos v + T' \cos E\right]$$

$$\frac{dS}{dt} = \frac{2\sqrt{S}\,(1 - S)\; r\, W' \cos \psi}{(\mu a)^{1/2} (1 - e^2)^{1/2}}$$

$$\frac{d\beta_3}{dt} = \frac{-r\, W' \sin \psi}{(\mu a S)^{1/2} (1 - e^2)^{1/2}}$$

$$\frac{d\beta_2}{dt} = \frac{1}{e\,(\mu a S)^{1/2} (1 - e^2)^{1/2}} \Big[a R' \sqrt{S}\,(1 - e^2) \cos v$$

$$- 2 T' a \sqrt{S}\,(1 - e \cos E) \sin v$$

$$- T' a e \sqrt{S}\,(1 - e \cos E) \cos v \sin v$$

$$+ a e (1 - e \cos E)\, W' \sqrt{1 - S} \sin \psi\Big]$$

$$\frac{d\beta_1}{dt} = \frac{\mu}{(-2\alpha_1)^{3/2}}\left[\frac{-2\, r\, R'}{(\mu a)^{1/2}} + (1 - e^2)^{1/2}\left(\frac{d\beta_2}{dt} + \sqrt{1 - S}\,\frac{d\beta_3}{dt}\right)\right]$$

$$+ \frac{3}{2}\,\frac{1}{a}\,\frac{da}{dt}\,(t - \beta_1), \tag{3.1}$$

where E, v , and ψ are the eccentric anomaly, true anomaly, and argument of latitude respectively.

The parameters R', T', and W' are defined as the components of total forces acting on the orbit in the radial, tangential and normal directions. In the spirit of the above discussion, we now consider that these force components are not real but virtual, that is, these forces are responsible for the 'energy fluxuations' or spread of the ergodic surfaces in phase space. In other words, the computed orbit has deviated from the true or observed orbit because of the presence of these virtual forces. It is our task to determine them, and once this is accomlished, the change in time of the six orbital elements can be obtained from equations (3.1), which in turn are then inserted into the Vinti orbit generator to produce corresponding changes in spacecraft coordinates, $\pm\Delta X$, $\pm\Delta Y$ and $\pm\Delta Z$. The plus or minus signs indicate a coordinate bound, or error bound in the ordinates as functions of time, since they derive from an error or uncertainty in the energy or Hamiltonian of the system.

At this point, we now assume that these errors reflected as virtual forces are described in terms of changes of the eccentric anomaly with time, that is,

$$\begin{aligned} R' &= R\,\dot{E} \\ T' &= T\,\dot{E} \\ W' &= W\,\dot{E} \end{aligned} \tag{3.2}$$

where R, T, and W are coefficients to be determined. Now Kepler's law states,

$$M = \hat{n}\,(t + \beta_1) = E - e \sin E, \tag{3.3}$$

where β_1 is the Vinti parameter related to the time of perigee passage. Differentiating,

$$\frac{dE}{dt} = \frac{n}{(1 - e\cos E)} = \frac{\mu^{1/2} a^{-3/2}}{(1 - e\cos E)} \tag{3.4}$$

Using this, équations (3.2) become,

$$R\,\dot{E} = \frac{R\,\mu^{1/2} a^{-3/2}}{(1 - e\cos E)} = \frac{d^2 q_R}{d t^2}$$

$$T\,\dot{E} = \frac{T\,\mu^{1/2} a^{-3/2}}{(1 - e\cos E)} = \frac{d^2 q_T}{d t^2} \tag{3.5}$$

$$W\,\dot{E} = \frac{W\,\mu^{1/2} a^{-3/2}}{(1 - e\cos E)} = \frac{d^2 q_W}{d t^2}$$

per unit mass, where the q_R, q_T, and q_W are taken to be displacements along the radial, tangential, and normal directions to the orbit. Therefore,

$$T = \frac{d^2 q_T}{d t^2}\;\frac{(1 - e\cos E)}{\mu^{1/2} a^{-3/2}}$$

$$R = \frac{d^2 q_R}{d t^2}\;\frac{(1 - e\cos E)}{\mu^{1/2} a^{-3/2}} \tag{3.6}$$

$$W = \frac{d^2 q_W}{d t^2}\;\frac{(1 - e\cos E)}{\mu^{1/2} a^{-3/2}}$$

In order to determine q_T, q_R, and q_W, we refer to Figures 3.1, 3.2, and 3.3.

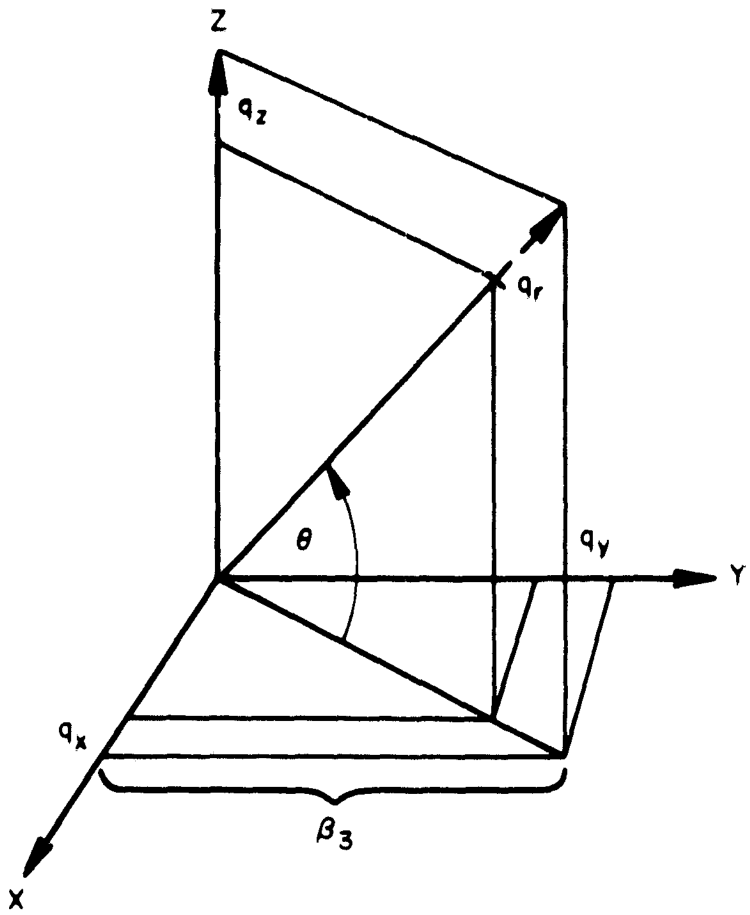


Figure 3.1

Here,

$$q_Z = q_R \sin \theta$$

$$q_X = q_R \cos \theta \cos \beta_3 \qquad (3.7)$$

$$q_Y = q_R \cos \theta \sin \beta_3$$

where θ is the geocentric latitude, and β_3 is the Vinti parameter associated with the geocentric longitude of the node.

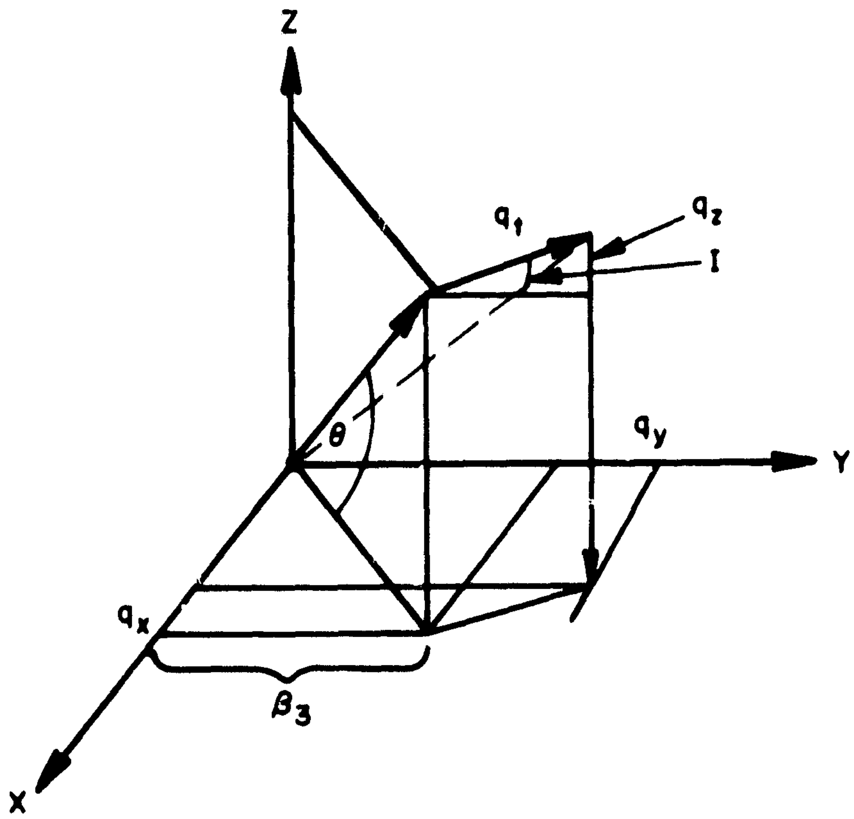


Figure 3.2

Here

$$q_Z = q_T \sin I$$

$$q_X = q_T \cos I \cos \left(\beta_3 + \frac{\pi}{2}\right) \qquad (3.8)$$

$$q_Y = q_T \cos I \sin \left(\beta_3 + \frac{\pi}{2}\right)$$

where I is the inclination of the orbital plane.

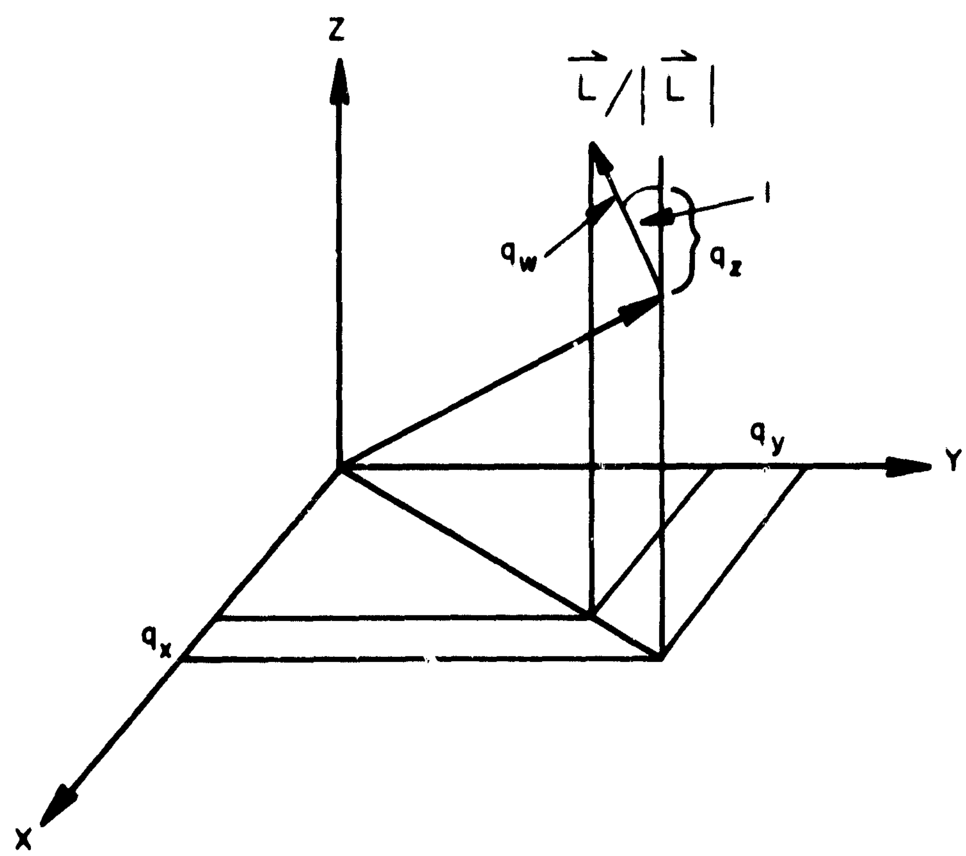


Figure 3.3

Here

$$q_Z = q_W \cos I$$
$$q_X = q_W \sin I \cos \beta_3 \qquad (3.9)$$
$$q_Y = q_W \sin I \sin \beta_3$$

Now summing all of the individual q_x, q_y, and q_z and identifying them as x, y, and z inertial respectively, we have,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} \cos\theta \cos\beta_3, & \cos I \cos\left(\frac{\pi}{2} + \beta_3\right), & \sin I \cos\beta_3 \\ \cos\theta \sin\beta_3, & \cos I \sin\left(\frac{\pi}{2} + \beta_3\right), & \sin I \sin\beta_3 \\ \sin\theta, & \sin I, & \cos I \end{pmatrix} \begin{pmatrix} q_R \\ q_T \\ q_W \end{pmatrix} \tag{3.10}$$

or simply,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{X} \begin{pmatrix} q_R \\ q_T \\ q_W \end{pmatrix} \tag{3.11}$$

Inverting,

$$\begin{pmatrix} q_R \\ q_T \\ q_W \end{pmatrix} = \mathbf{X}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{3.12}$$

Since the inertial coordinates are known functions of the orbital elements (Reference 5), we have that the coordinates q_R, q_T and q_W are also. As a result, Equations (3.6) can then be expressed in the form,

$$T = \frac{(1 - e\cos F)}{\mu^{1/2} a^{-3/2}} \left\{ \left(\frac{\partial^2 q_T}{\partial a^2}\right) \frac{da}{dt} \cdot \left(\frac{d a^2}{d t^2}\right) \frac{\partial q_T}{\partial a} + \cdots + \left(\frac{\partial^2 q_T}{\partial \beta_3^2}\right) \frac{d\beta_3}{dt} + \left(\frac{d^2 \beta_3}{d t^2}\right) \frac{\partial q_T}{\partial \beta_3} \right\}$$

$$R = \frac{(1 - e \cos E)}{\mu^{1/2} a^{-3/2}} \left\{ \left(\frac{\partial^2 q_R}{\partial a^2} \right) \frac{d a}{d t} + \left(\frac{d^2 a}{d t^2} \right) \frac{\partial q_R}{\partial a} + \cdots \right\}$$

$$W = \frac{(1 - e \cos E)}{\mu^{1/2} a^{-3/2}} \left\{ \left(\frac{\partial^2 q_W}{\partial a^2} \right) \frac{d a}{d t} + \left(\frac{d^2 a}{d t^2} \right) \frac{\partial q_W}{\partial a} + \cdots \right\} \tag{3.13}$$

The right hand sides of equations (3.13) are functions of R, T, and W, so that we have,

$$\begin{aligned} T &= f\,(R,\ T,\ W) \\ R &= g\,(R,\ T,\ W) \\ W &= h\,(R,\ T,\ W) \end{aligned} \tag{3.14}$$

a system of transcendental equations where the unknowns are located on both the left and right hand sides. By examining equations (3.13) it is seen that each quantity is known in terms of R, T and W. For example, substituting equations (3.2) into (3.1) we have,

$$\frac{d a}{d t} = 2 \left(\frac{a^3}{\mu} \right)^{1/2} (1 - e^2)^{-1/2} \,[R\, e \sin v + T + T\, e \cos v]\, \frac{d E}{d t}$$

$$\frac{d e}{d t} = \left(\frac{a}{\mu} \right)^{1/2} \sqrt{1 - e^2} \left[R \frac{\sqrt{1 - e^2} \cos E}{1 - e \cos E} \right.$$

$$\left. + T \cos E + T \frac{(\cos E - e)}{1 - e \cos E} \right] \frac{d E}{d t}$$

$$\frac{d S}{d t} = \frac{2 \sqrt{S\,(1 - S)}\; r\, W \cos (v + \beta_2)}{(\mu a)^{1/2} \sqrt{1 - e^2}} \; \frac{d E}{d t}$$

$$\frac{d\beta_3}{dt} = \frac{-a(1-e\cos E)}{\sqrt{\mu a S(1-e^2)}}\left[\sin v\cos\beta_2 + \cos v\sin\beta_2\right]\frac{dE}{dt}$$

$$\frac{d\alpha_1}{dt} = \frac{-\mu}{(-2\alpha_1)^{3/2}}\left[\frac{-2rR}{(\mu a)^{1/2}}\frac{dE}{dt}\right.$$

$$\left. + \sqrt{1-e^2}\left(\frac{d\beta_2}{dt} + \sqrt{1-S}\,\frac{d\beta_2}{dt}\right)\right] - \frac{3}{2a}\frac{da}{dt}(t-\beta_1)$$

$$\frac{d\beta_2}{dt} = \frac{1}{e\sqrt{\mu a S(1-e^2)}}\left[\sqrt{S}\,a R(1-e^2)\cos v\right.$$

$$-2Ta\sqrt{S}\sin v + 2Tae\sqrt{S}\cos E\sin v$$

$$-Tae\sqrt{S}\cos v\sin v + Tae^2\sqrt{S}\cos E\cos v\sin v$$

$$+ae\sqrt{1-S}\,W\cos\beta_2\sin v + aeW\sqrt{1-S}\sin\beta_2\cos v$$

$$\left. -ae^2\sqrt{1-S}\,W\cos\beta_2\sin v - ae^2\sqrt{1-S}\,W\sin\beta_2\cos v\right]\frac{dE}{dt} \tag{3.15}$$

Differentiating again with respect to time we have,

$$\frac{d^2a}{dt^2} = \frac{2e}{(1-e\cos E)^3}\left\{\frac{\mu^{1/2}a^{-3/2}}{(1-e\cos E)}\left[R(\cos E - e)\right.\right.$$

$$\left. -T\sqrt{1-e^2}\sin E\right] - \frac{\sin E}{\sqrt{1-e^2}}\left[eR\sqrt{1-e^2}\sin E\right.$$

$$\left.\left. + T + Te(\cos E - e)\right]\right\}$$

$$\frac{d^2 e}{d t^2} = \frac{\mu^{1/2} \; 1 - e^2}{a\,(1 - e \cos E)^3} \left\{ \frac{1}{a^{3/2}\,(1 - e \cos E)} \left[R \sqrt{1 - e^2} \cos E \right.\right.$$

$$+ (1 - e^2)\, T \sin E - T \sin E\,(1 - e \cos E)^2 \Big]$$

$$- e \sin E \left[R \sqrt{1 - e^2} \sin E + T\,(\cos E - e)\right.$$

$$\left.\left. + T \cos E\,(1 - e \cos E)\right]\right\}$$

$$\frac{d^2 S}{d t^2} = \frac{2 \sqrt{S\,(1 - S)}\; W \sqrt{1 - e^2}}{a\,(1 - e \cos E)} \left\{ \frac{\mu^{1/2}}{(1 - e \cos E)\; a^{3/2}} \left[e \sin E \cos \psi \right.\right.$$

$$- \frac{1}{(1 - e \cos E)} \left((1 - e^2) \sin E \cos \beta_2 \right.$$

$$\left.\left. + (1 - e^2)^{1/2} (\cos F - e) \sin \beta_2 \right) \right]$$

$$\left. - e \cos \psi \sin E \right\}$$

$$\frac{d^2 \beta_3}{d t^2} = \frac{1}{a \sqrt{S\,(1 - e^2)}\;(1 - e \cos E)} \left\{ e \sin F \sin \psi \left[1 - \frac{\mu^{1/2}}{(1 - e \cos E)}\right]\right.$$

$$\left. - \mu^{1/2} \frac{\left[\cos \beta_2 \sqrt{1 - e^2}\,(\cos E - e) - (1 - e^2) \sin \beta_2 \sin E\right]}{a^{3/2}\,(1 - e \cos E)^2} \right\}$$

$$\frac{d^2 \beta_1}{d t^2} = - \mu^{-1/2}\, a^{3/2} \left\{ \sqrt{1 - e^2} \left(\frac{d^2 \beta_2}{d t^2} + \sqrt{1 - S^2}\; \frac{d^2 \beta_3}{d t^2} \right) \right.$$

$$+\frac{2\,e\,R\sin E}{a\,(1-e\cos E)}\left[1-\mu^{1/2}\,a^{-3/2}\right]\Bigg\}$$

$$-\frac{3}{2a}\left[(t-\beta_1)\frac{d^2 a}{d t^2}+\frac{d a}{d t}\right]$$

$$\frac{d^2\beta_2}{d t^2}=\frac{1}{e\sqrt{\mu\,a\,S}\,(1-e^2)}\left\{\frac{-a\,R\sqrt{S}\,(1-e^2)^2\sin E}{(1-e\cos E)^2}\right.$$

$$-\frac{2\,T\,e\,a\sqrt{S}\,(\cos E-e)\sqrt{1-e^2}}{(1-e\cos E)^2}$$

$$+\frac{2\,T\,a\,e\sqrt{S}\,(\cos E-e)\cos E\sqrt{1-e^2}}{(1-e\cos E)}$$

$$-\frac{2\,T\sqrt{S}\,a\,e\,(1-e^2)\sin^2 E}{(1-e\cos E)}$$

$$-\frac{T\,a e\sqrt{S}\,(1-e^2)^{1/2}(\cos E-e)^2}{(1-e\cos E)^3}$$

$$+\frac{T\,a\,e\sqrt{S}\,(1-e^2)\sin^2 E}{(1-e\cos E)^3}+\frac{T\,a\,e^2\sqrt{S}\cos E\,(\cos E-e)^2\sqrt{1-e^2}}{(1-e\cos E)^3}$$

$$+\frac{T\,a\,e^2\sqrt{S}\cos E\sin^2 E\,(1-e^2)^{3/2}}{(1-e\cos E)^3}$$

$$-\frac{T\,a\,e^2\sqrt{S}\sin^2 E\,(\cos E-e)\sqrt{1-e^2}}{(1-e\cos E)^2}$$

$$+ \frac{a e \sqrt{1 - S} W \cos \beta_2 \sqrt{1 - e^2} (\cos E - e)}{(1 - e \cos E)^2}$$

$$- \frac{a e \sqrt{1 - S} W \sin \beta_2 \sin E (1 - e^2)}{(1 - e \cos E)^2}$$

$$- \frac{a e^2 \sqrt{1 - S} W \cos \beta_2 \sqrt{1 - e^2} (\cos E - e)}{(1 - e \cos E)^2}$$

$$\left. + \frac{a e^2 \sqrt{1 - S} W \sin \beta_2 \sin E (1 - e^2)}{(1 - e \cos E)^2} \right\}$$

$$- \frac{\sin E}{a^2 (1 - e \cos E)^2 \sqrt{S (1 - e^2)}} \left\{ \frac{- a R \sqrt{S} (1 - e^2)^2 \sin E}{(1 - e \cos E)^2} \right.$$

$$- \frac{2 T \sqrt{S} a \sqrt{1 - e^2} \sin E}{(1 - e \cos E)} + \frac{2 T \sqrt{S} a e \sqrt{1 - e^2} \cos E \sin E}{(1 - e \cos E)}$$

$$- \frac{T a e \sqrt{S} \sin E (\cos E - e) \sqrt{1 - e^2}}{(1 - e \cos E)^2}$$

$$+ \frac{T a e^2 \sqrt{S} \cos E \sin E (\cos E - e)}{(1 - e \cos E)^2}$$

$$+ \frac{a e \sqrt{1 - S} W \cos \beta_2 \sin E \sqrt{1 - e^2}}{(1 - e \cos E)}$$

$$+ \frac{a e \sqrt{1 - S} W \sin \beta_2 (\cos E - e)}{(1 - e \cos E)}$$

$$- \frac{a e^2 \sqrt{1 - S}\, W \cos \beta_2 \sin E \sqrt{1 - e^2}}{(1 - e \cos E)}$$

$$\left. - \frac{a e^2 \sqrt{1 - S}\, W \sin \beta_2 (\cos E - e)}{(1 - e \cos E)} \right\} \tag{3.16}$$

From equations (3.12) we have that,

$$\begin{pmatrix} \dfrac{\partial q_R}{\partial \ell} \\ \\ \dfrac{\partial q_T}{\partial \ell} \\ \\ \dfrac{\partial q_W}{\partial \ell} \end{pmatrix} = X^{-1} \begin{pmatrix} \dfrac{\partial x}{\partial \ell} \\ \\ \dfrac{\partial y}{\partial \ell} \\ \\ \dfrac{\partial z}{\partial \ell} \end{pmatrix} \tag{3.17}$$

where $\ell = 1, \ldots, 6$ denotes a, e, . . . , β_3. Here $\partial X/\partial \ell$, $\partial y/\partial \ell$, and $\partial z/\partial \ell$ are given in Reference 6, or if one wishes to bypass calculation of the right ascension, by differentiating equations (15), (16), and (17) of page 34 of Reference 5, which give,

$$\frac{\partial X}{\partial \ell_i} = \frac{X \rho}{\rho^2 + c^2} \frac{\partial \rho}{\partial \ell_i} + (\rho^2 + c^2)^{1/2} \left\{ - H_1 \frac{\partial \psi}{\partial \ell_i} \right.$$

$$\times \left[\cos \Omega \sin \psi + \operatorname{sgn}(\alpha_3) H_3 (1 - S)^{1/2} \cos \psi \frac{\sin \Omega}{H_1^2} \right]$$

$$+ \frac{\partial H_1}{\partial \ell_i} \left[\cos \Omega \cos \psi + \sin \Omega \operatorname{sgn}(\alpha_3) (1 - S)^{1/2} \right.$$

$$\left. \times (H_2 + H_3 \sin \psi) \frac{1}{H_1^2} \right] - \sin \Omega \operatorname{sgn}(\alpha_3) (1 - S)^{1/2}$$

$$\times \frac{1}{H_1}\left[\frac{\partial H_2}{\partial \ell_i} + \sin\psi \frac{\partial H_3}{\partial \ell_i}\right] - H_1 \frac{\partial \Omega}{\partial \ell_i}\left[\cos\psi \sin\Omega\right.$$

$$\left. + \operatorname{sgn}(\alpha_3)(1 - S)^{1/2}(H_2 + H_3 \sin\psi)\frac{1}{H_1^2}\cos\Omega\right]$$

$$\left. + \delta_{3i}\frac{1}{2}\operatorname{sgn}(\alpha_3)\frac{(H_2 + H_3 \sin\psi)}{H_1 (1 - S)^{1/2}}\sin\Omega\right\}$$

$$\frac{\partial y}{\partial \ell_i} = y \frac{\rho}{\rho^2 + c^2}\frac{\partial \rho}{\partial \ell_i} + (\rho^2 + c^2)^{1/2}\left\{\cdot H_1 \frac{\partial \psi}{\partial \ell_i}\right.$$

$$\times \left[\sin\Omega \sin\psi - \operatorname{sgn}(\alpha_3) H_3 (1 - S)^{1/2}\frac{\cos\psi}{H_1^2}\cos\Omega\right]$$

$$+ \frac{\partial H_1}{\partial \ell_i}\left[\sin\Omega \cos\psi - \cos\Omega \operatorname{sgn}(\alpha_3)(1 - S)^{1/2}\right.$$

$$\left.\times (H_2 + H_3 \sin\psi)\frac{1}{H_1^2}\right] + \frac{\operatorname{sgn}(\alpha_3)(1 - S)^{1/2}}{H_1}\cos\Omega$$

$$\times \left[\frac{\partial H_2}{\partial \ell_i} + \sin\psi \frac{\partial H_3}{\partial \ell_i}\right] + H_1 \frac{\partial \Omega}{\partial \ell_i}\left[\cos\Omega \sin\psi\right.$$

$$\left. - \sin\Omega \operatorname{sgn}(\alpha_3)(1 - S)^{1/2}(H_2 + H_3 \sin\psi)\frac{1}{H_1^2}\right]$$

$$\left. - \delta_{3i}\frac{1}{2}\operatorname{sgn}(\alpha_3)\frac{(H_2 + H_3 \sin\psi)}{H_1 (1 - S)^{1/2}}\cos\Omega\right\}$$

and

$$\frac{\partial z}{\partial \ell_i} = \rho \frac{\partial \eta}{\partial \ell_i} + \eta \frac{\partial \rho}{\partial \ell_i} \tag{3.17}$$

where i = 1, 2, 3, and δ_{3i} = 0 for i = 1, 2, and one for i = 3. For i = 4, 5,

$$\begin{aligned}
\frac{\partial X}{\partial \ell_i} = X \frac{\rho}{\rho^2 + c^2} \frac{\partial \rho}{\partial \ell_i} + (\rho^2 + c^2)^{1/2} \Bigg\{ & - H_1 \frac{\partial \psi}{\partial \ell_i} \\
& \times \Big[\cos \Omega \sin \psi + \operatorname{sgn}(\alpha_3) H_3 (1 - S)^{1/2} \\
& \times \cos \psi \frac{1}{H_1^2} \sin \Omega\Big] - H_1 \frac{\partial \Omega}{\partial \ell_i} \Big[\sin \Omega \cos \psi \\
& + \operatorname{sgn}(\alpha_3) (1 - S)^{1/2} \left(H_2 + H_3 \frac{\sin \psi}{H_1^2} \cos \Omega\right)\Big]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial y}{\partial \ell_i} = y \frac{\rho}{\rho^2 + c^2} \frac{\partial \rho}{\partial \ell_i} + (\rho^2 + c^2)^{1/2} \Bigg\{ & - H_1 \frac{\partial \psi}{\partial \ell_i} \\
& \times \Big[\sin \Omega \sin \psi - \operatorname{sgn}(\alpha_3) H_3 (1 - S)^{1/2} \\
& \times \cos \psi \frac{1}{H_1^2} \cos \Omega\Big] + H_1 \frac{\partial \Omega}{\partial \ell_i} \Big[\cos \Omega \cos \psi \\
& - \operatorname{sgn}(\alpha_3) (1 - S)^{1/2} \left(H_2 + H_3 \frac{\operatorname{sgn} \psi}{H_1^2}\right) \cos \Omega\Big]\Bigg\}
\end{aligned}$$

and

$$\frac{\partial Z}{\partial \ell_i} = \rho \frac{\partial \eta}{\partial \ell_i} + \eta \frac{\partial \rho}{\partial \ell_i} . \tag{3.18}$$

For i = 6,

$$\frac{\partial X}{\partial \ell_6} = -y, \quad \frac{\partial Y}{\partial \ell_6} = X, \quad \frac{\partial Z}{\partial \ell_6} = 0 \tag{3.19}$$

To complete the right hand side of equations (3.13), we now can obtain $\partial^2 x/\partial \ell_i^2$, $\partial^2 y/\partial \ell_i^2$ and $\partial^2 z/\partial \ell_i^2$ by either differentiating the first partials given by Reference 6, or by differentiating equations (3.17) through (3.19) again. For example, from (3.19) we have,

$$\frac{\partial^2 X}{\partial \ell_6^2} = -x, \quad \frac{\partial^2 y}{\partial \ell_6^2} = -y, \quad \frac{\partial^2 z}{\partial \ell_6^2} = 0 \tag{3.20}$$

For i = 1, 2, 3,

$$\frac{\partial^2 X}{\partial \ell_i^2} = \frac{\partial X}{\partial \ell_i} \frac{\rho}{\rho^2 + c^2} \frac{\partial \rho}{\partial \ell_i} + \frac{x \rho}{\rho^2 + c^2} \frac{\partial^2 \rho}{\partial \ell_i^2}$$

$$+ \frac{x}{(\rho^2 + c^2)} \left(\frac{\partial \rho}{\partial \ell_i}\right)^2 (1 + 2\rho^2) + \frac{\rho}{(\rho^2 + c^2)^{1/2}} \left\{ - H_1 \frac{\partial \psi}{\partial \ell_i} \right.$$

$$\times \left[\cos \Omega \sin \psi + \operatorname{sgn}(\alpha_3) H_3 (1 - S)^{1/2} \frac{\cos \psi}{H_1^2} \sin \Omega\right]$$

$$+ \frac{\partial H_1}{\partial \ell_i} \left[\cos \Omega \cos \psi + \sin \Omega \operatorname{sgn}(\alpha_3) (1 - S)^{1/2}\right.$$

$$\left. \times (H_2 + H_3 \sin \psi) \frac{1}{H_1^2}\right] - \sin \Omega \operatorname{sgn}(\alpha_3) (1 - S)^{1/2}$$

$$\times \frac{1}{H_1} \left[\frac{\partial H_2}{\partial \ell_i} + \sin \psi \frac{\partial H_3}{\partial \ell_i}\right] - H_1 \frac{\partial \Omega}{\partial \ell_1} \left[\cos \psi \sin \Omega\right.$$

$$+ \operatorname{sgn}(\alpha_3)(1 - S)^{1/2}(H_2 + H_3 \sin\psi)\frac{1}{H_1^2}\cos\psi\Big]$$

$$+ c_{3i}\frac{1}{2}\operatorname{sgn}(\alpha_2)\frac{(H_2 + H_3 \sin\psi)}{H_1(1 - S)^{1/2}}\sin\psi\Big\}$$

$$+ (\mu^2 + c^2)^{1/2}\Big\{ H_i \frac{\partial^2 \psi}{\partial \ell_i^2}\Big[\cos\Omega \sin\psi + \operatorname{sgn}(\alpha_3)$$

$$\times H_3 (1 - S)^{1/2}\cos\psi \frac{1}{H_1^2}\sin\Omega\Big] - H_1 \frac{\partial\psi}{\partial \ell_i}\Big[\cos\psi\cos\Omega\frac{\partial\psi}{\partial \ell_i}$$

$$- \frac{\partial H_1}{\partial \ell_i}\frac{1}{H_1^3}\operatorname{sgn}(\alpha_3) H_3 (1 - S)^{1/2}\cos\psi\sin\Omega$$

$$- \sin\psi\sin\Omega\frac{\partial\Omega}{\partial \ell_i} + \operatorname{sgn}(\alpha_3)\frac{H_3}{H_1^2}(1 - S)^{1/2}$$

$$\times \cos\psi\cos\Omega\frac{\partial\Omega}{\partial \ell_i} - \operatorname{sgn}(\alpha_3)\frac{H_3}{H_1^2}(1 - S)^{1/2}$$

$$\times \sin\Omega\sin\psi\frac{\partial\psi}{\partial \ell_i}\Big] + \frac{\partial^2 H_1}{\partial \ell_i^2}\Big[\cos\Omega\cos\psi$$

$$+ \sin\Omega\operatorname{sgn}(\alpha_3)(1 - S)^{1/2}(H_2 + H_3 \sin\psi)\frac{1}{H_1^2}\Big]$$

$$+ \frac{\partial H_i}{\partial \ell_i}\Big[- \cos\Omega\sin\psi\frac{\partial\psi}{\partial \ell_i} - \sin\Omega\cos\psi\frac{\partial\Omega}{\partial \ell_i}$$

$$+ \cos \Omega \frac{\partial \Omega}{\partial \ell_i} \operatorname{sgn}(\alpha_3) (1 - S)^{1/2} (H_2 + H_3 \sin \psi) \frac{1}{H_1^2}$$

$$+ \sin \Omega \operatorname{sgn}(\alpha_3) (1 - S)^{1/2} \left(\frac{1}{H_1^2} \left(\frac{\partial H_2}{\partial \ell_i} + \sin \psi \frac{\partial H_3}{\partial \ell_i} \right.\right.$$

$$\left.\left. + H_3 \cos \psi \frac{\partial \psi}{\partial \ell_i} \right)\right) - 2 \frac{(H_2 + H_3 \sin \psi)}{H_1^3} \left.\frac{\partial H_1}{\partial \ell_i}\right]$$

$$- \cos \Omega \frac{\partial \Omega}{\partial \ell_i} \operatorname{sgn}(\alpha_3) (1 - S)^{1/2} \frac{1}{H_1} \left[\frac{\partial H_2}{\partial \ell_i} + \sin \psi \frac{\partial H_3}{\partial \ell_i} \right]$$

$$- \sin \Omega \operatorname{sgn}(\alpha_3) (1 - S)^{1/2} - \frac{1}{H_1^2} \frac{\partial H_1}{\partial \ell_i} \left[\frac{\partial H_2}{\partial \ell_i} + \sin \psi \frac{\partial H_3}{\partial \ell_i} \right]$$

$$+ \frac{1}{H_1} \left[\frac{\partial^2 H_2}{\partial \ell_i^2} + \cos \psi \frac{\partial \psi}{\partial \ell_i} \frac{\partial H_3}{\partial \ell_i} + \sin \psi \frac{\partial^2 H_3}{\partial \ell_i^2} \right]$$

$$- \left(H_1 \frac{\partial^2 \Omega}{\partial \ell_i^2} + \frac{\partial H_1}{\partial \ell_i} \frac{\partial \Omega}{\partial \ell_i} \right) \left[\cos \psi \sin \Omega + \operatorname{sgn}(\alpha_3) (1 - S)^{1/2} \right.$$

$$\left. \times (H_2 + H_2 \sin \psi) \frac{1}{H_1^2} \cos \Omega \right]$$

$$- H_1 \frac{\partial \Omega}{\partial \ell_i} \left[\cos \psi \cos \Omega \frac{\partial \Omega}{\partial \ell_i} - \sin \Omega \sin \Omega \frac{\partial \psi}{\partial \ell_i} \right.$$

$$+ \operatorname{sgn}(\alpha_3) (1 - S)^{1/2} \left(\left(\frac{\partial H_2}{\partial \ell_i} + \frac{\partial H_3}{\partial \ell_i} \sin \psi + H_3 \cos \psi \frac{\partial \psi}{\partial \ell_i} \right) \right.$$

$$\times \frac{1}{H_1^2} \cos \Omega - \frac{(H_2 + H_3 \sin \psi)}{H_1^2} \sin \Omega \frac{\partial \Omega}{\partial \ell_i}$$

$$- \frac{2}{H_1^3} \frac{\partial H_1}{\partial \ell_i} \cos \Omega (H_2 + H_3 \sin \psi)\Bigg)\Bigg]$$

$$+ \delta_{3i} \frac{\operatorname{sgn}(\alpha_3)}{2(1 - S)^{1/2}} \Bigg[\cos \Omega \frac{\partial \Omega}{\partial \ell_i} \frac{1}{H_1} (H_2 + H_3 \sin \psi)$$

$$+ \frac{\sin \Omega}{H_1} \left(\frac{\partial H_2}{\partial \ell_i} + \cos \psi \frac{\partial \psi}{\partial \ell_i} H_3 + \sin \psi \frac{\partial H_3}{\partial \ell_i} \right)$$

$$- \frac{1}{H_1^2} \frac{\partial H_1}{\partial \ell_i} \sin \Omega (H_2 + H_3 \sin \psi) \Bigg], \qquad (3.21)$$

and similarly for $\partial^2 y/\partial \ell_i^2$, and $\partial^2 z/\partial \ell_i^2$. For i = 4, 5, we have from equations (3.18),

$$\frac{\partial^2 x}{\partial \ell_i^2} = \frac{\partial x}{\partial \ell_i} \frac{\rho}{\rho^2 + c^2} \frac{\partial \rho}{\partial \ell_i} + \frac{x \rho}{\rho^2 + c^2} \frac{\partial^2 \rho}{\partial \ell_i^2}$$

$$+ \frac{x}{(\rho^2 + c^2)} \left(\frac{\partial \rho}{\partial \ell_i}\right)^2 (1 + 2 \rho^2) + \frac{\rho}{(\rho^2 + c^2)^{1/2}} \Bigg\{ - H_1 \frac{\partial \psi}{\partial \ell_i}$$

$$\times \left[\cos \Omega \sin \psi + \operatorname{sgn}(\alpha_3) H_3 (1 - S)^{1/2} \frac{\cos \psi}{H_1^2} \sin \Omega \right]$$

$$- H_1 \frac{\partial \Omega}{\partial \ell_i} \Bigg[\sin \Omega \cos \psi + \operatorname{sgn}(\alpha_3) (1 - S)^{1/2}$$

$$\times (H_2 + H_3 \sin\psi) \frac{1}{H_1^2} \cos\Omega \Big]\Big\}$$

$$+ (\rho^2 + c^2)^{1/2} \Big\{ - H_1 \frac{\partial^2 \psi}{\partial \ell_i^2} \Big[\cos\Omega \sin\psi + \mathrm{sgn}\,(\alpha_3)$$

$$\times H_3 (1 - S)^{1/2} \cos\psi \frac{1}{H_1^2} \sin\Omega \Big] - H_1 \frac{\partial \psi}{\partial \ell_i} \Big[\cos\psi$$

$$\times \cos\Omega \frac{\partial \Omega}{\partial \ell_i} - \sin\psi \sin\Omega \frac{\partial \Omega}{\partial \ell_i}$$

$$+ \mathrm{sgn}\,(\alpha_3) \frac{H_3}{H_1^2} (1 - S)^{1/2} \cos\psi \cos\Omega \frac{\partial \Omega}{\partial \ell_i}$$

$$- \mathrm{sgn}\,(\alpha_3) \frac{H_3}{H_1^2} (1 - S)^{1/2} \sin\Omega \sin\psi \frac{\partial \psi}{\partial \ell_i}$$

$$- \frac{\partial H_1}{\partial \ell_i} \frac{1}{H_1^3} \mathrm{sgn}\,(\alpha_3) H_3 (1 - S)^{1/2} \cos\psi \sin\Omega \Big]$$

$$- \frac{\partial H_1}{\partial \ell_i} \frac{\partial \Omega}{\partial \ell_i} \Big[\sin\Omega \cos\psi + \mathrm{sgn}\,(\alpha_3) (1 - S)^{1/2}$$

$$\times (H_2 + H_3 \sin\psi) \frac{1}{H_1^2} \cos\Omega \Big]$$

$$- H_1 \frac{\partial^2 \Omega}{\partial \ell_i^2} \Big[\sin\Omega \cos\psi + \mathrm{sgn}\,(\alpha_3) (1 - S)^{1/2}$$

$$\times (H_2 + H_3 \sin\psi) \frac{1}{H_1^2} \cos\Omega \Big]$$

$$- H_1 \frac{\partial\Omega}{\partial \ell_i} \left[\cos\Omega \sin\psi \frac{\partial\Omega}{\partial \ell_i} - \sin\Omega \sin\psi \frac{\partial\psi}{\partial \ell_i} \right.$$

$$+ \operatorname{sgn}(\alpha_3)(1 - S)^{1/2} \left(- \sin\Omega \frac{\partial\Omega}{\partial \ell_i} \frac{(H_2 + H_3 \sin\psi)}{H_1^2} \right.$$

$$+ \frac{\cos\Omega}{H_1^2} \left(\frac{\partial H_2}{\partial \ell_i} + \frac{\partial H_3}{\partial \ell_i} \sin\psi + \cos\psi \frac{\partial\psi}{\partial \ell_i} H_3 \right)$$

$$\left. \left. - 2 \frac{(H_2 + H_3 \sin\psi)}{H_1^3} \cos\Omega \frac{\partial H_1}{\partial \ell_i} \right) \right] \qquad (3.22)$$

and similarly for $\partial^2 y/\partial \ell_i^2$ and $\partial^2 z/\partial \ell_i^2$. The method for computing the coordinate error bounds can now be defined, since the right hand side of equations (3.13) are known in terms of R, T, and W.

IV. ALGORITHM

The procedure for computing coordinate bounds is as follows:

1. Enter into equations (3.15) an estimate for R,T, and W based on spacecraft geometry and nominal orbit conditions, in units of Newtons-force. From past studies, a good initial estimate for a Vinti solution of a Relay II type orbit would be about 10^{-7} Newtons for R and W, and approximately 10^{-6} Newtons for T since the tangential forces and in-track position errors are usually larger (References 1).

2. The corresponding $\Delta a, \Delta e, \ldots, \Delta \beta_3$ are used in the Vinti orbit generator to produce equivalent coordinate differences ΔX_i, ΔY_i, and ΔZ_i, where i refers to the observation.

3. For any given tracking data such as radio direction cosine for example, a corresponding ΔL_i and ΔM_i can be determined, which also can be called computed residuals.

4. If the 'true' residuals obtained by comparing theory with each observation are denoted as R_L and R_M, then the standard deviation of fit is given by

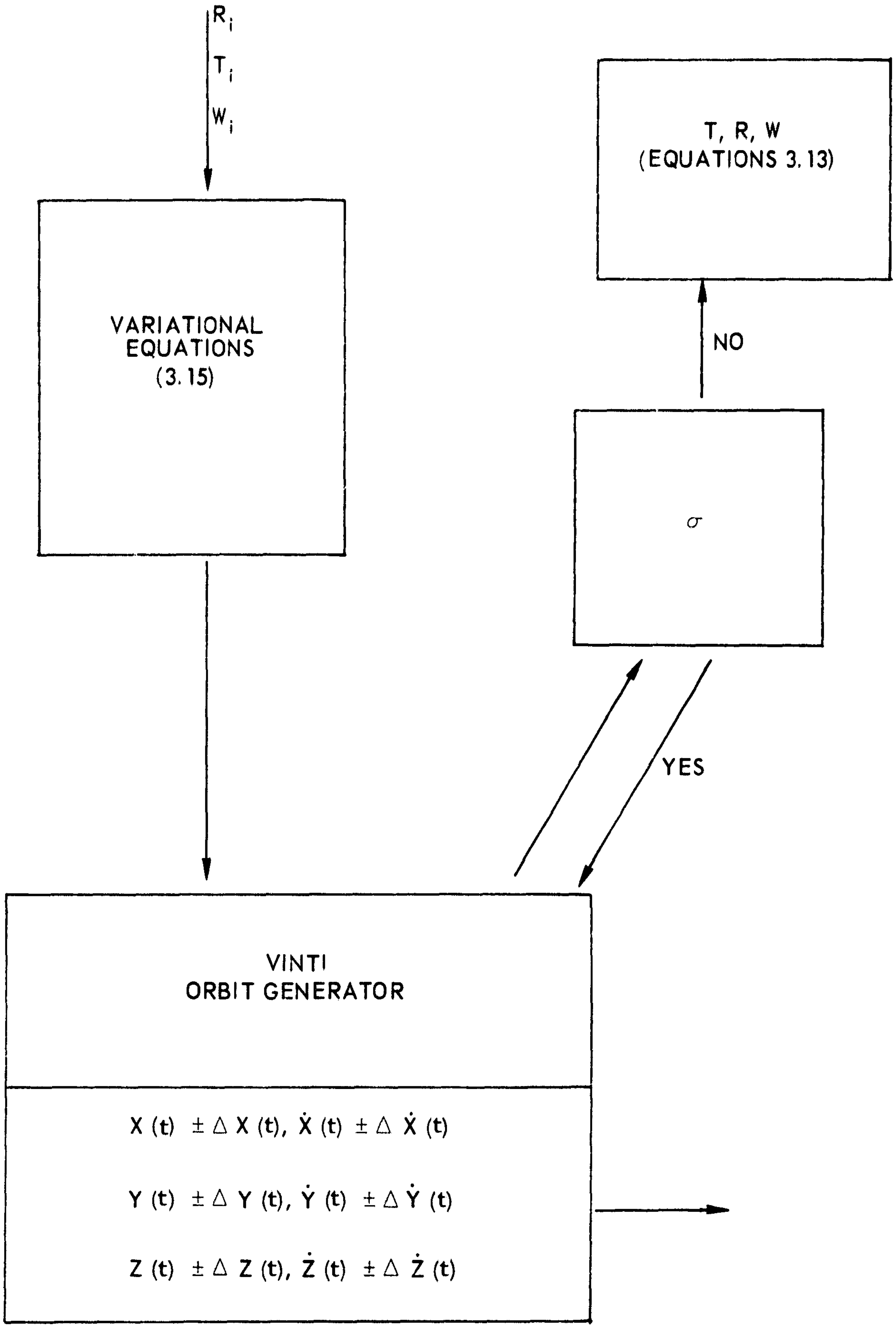
$$\sigma = \sqrt{\frac{1}{2n-6} \sum_{i=1}^{n} [(\Delta L_i)^2 + (\Delta M_i)^2]} \tag{4.1}$$

where ΔL_i and ΔM_i are the differences between true and computed residuals, or $(\Delta L_i - R_L)$ and $(\Delta M_i - R_M)$ respectively.

5. We wish σ to be arbitrarily close to zero. If our criterion is not met, we return to equations (3.13), insert our initial estimates for R, T, and W into the right hand sides and compute a new set of R, T, and W on the left. These new or iterated values are entered in the equations (3.15) again, and steps (2) through (4) are repeated.

6. This process is continued until a self consistent solution is found for the system of equations (3.13) which allows σ to be smaller than some preassigned, arbitrary, positive number. It is these values of R, T, and W that are accepted and used in conjunction with the set of equations (3.15) together with the Vinti orbit generator to then produce a set of corresponding error bounds as functions of time, that are given as $X(t) \pm \Delta X(t)$, $Y(t) \pm \Delta Y(t)$, and $Z(t) \pm \Delta Z(t)$.

V. FLOW DIAGRAM



VI. CONCLUSIONS

Equation (2.3) can now be considered as follows: Although an integrating anomaly equivalent to temperature is not immediately clear for this classical system, the most likely or expectation value for a coordinate will depend upon the form of the Hamiltonian in the ensemble. As a result, if the probability density ρ contains the Vinti Hamiltonian, then the most likely values of the spacecraft coordinates are those generated by the Vinti program. In addition, the size of the error volume $(\pm\Delta X)$ $(\pm\Delta Y)$ $(\pm\Delta Z)$ or $2|\Delta X||\Delta Y||\Delta Z|$ that bounds these most likely values are also determined by the Hamiltonian. If all interactions were known exactly, then obviously, the size of the cube or error volume would shrink to a point, namely that determined by the Hamiltonian, and the most likely values would then be the exact values for all time.

This same analysis, it is felt, may be applied to spacecraft trajectory systems in the pre-launch phase. A set of virtual forces R, T, and W can be determined by selecting and applying tracking data in this program particular to certain classes of orbits. The error bounds or volume produced can then be classified as nominal.

The interest at present is to interface this program with the existing Vinti orbit determination system and to apply it to several satellites of current interest.

VII. ACKNOWLEDGMENTS

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